# ON IMPROVING THE CONVERGENCE OF SERIES USED IN SOLVING THE HEAT-CONDUCTION EQUATION

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A method of improving the convergence of series used in solving heat-conduction equations by the method of finite integral transformations is examined. The method constitutes an extension of G. A. Grinberg's procedure to heat-conduction problems.

A great deal of attention has recently been paid to questions of using finite integral transformations in problems of transient heat conduction [1, 2]. A. V. Lykov [2] set out the advantages of this method in detail and also noted one serious disadvantage, namely, the nonuniform convergence of the series obtained as a result of the solution at the boundaries of the ranges of variation of the transformation variable.

One method of eliminating this shortcoming was proposed in [3]; it was discussed in detail and further developed in [2]. The essence of the process amounts to the solution of an auxiliary quasi-steady-state heat-conduction problem with corresponding boundary conditions.

In this paper we shall consider another method of improving the convergence of the series, not involving the solution of auxiliary heat-conduction boundary problems, but enabling us to improve the convergence of solutions obtained by the method of finite integral transformations in a very simple and physically-transparent manner. This procedure amounts to an extension of G. A. Grinberg's method [4] to heat-conduction problems.

It was shown in [4] that the reason for the poor convergence of the series constituting the solutions of boundary problems with inhomogeneous boundary conditions obtained by the method under consideration lay in the fact that the series were incapable of satisfying inhomogeneous boundary conditions, whereas individual terms in these series did satisfy homogeneous conditions. Hence in the neighborhood of the interval boundaries the series converged in a nonuniform manner.

Let the solution to the linear heat-conduction problem obtained by the method of finite integral transformations be known:

$$t(x,\ \tau) = \sum_{n=1}^{\infty} \bar{t}_n(\tau) K_n(x). \tag{1}$$

Let us set up an expression of the form

$$t(x,\ \tau) = \left[\sum_{n=1}^{\infty} \bar{t}_n(\tau) K_n(x) - V(x,\ \tau)\right] + V(x,\ \tau), \tag{2}$$

where $V(x,\ \tau)$ is a certain function satisfying the same boundary conditions as the solution (1).

If then we can expand the function $V(x,\ \tau)$ in a generalized Fourier series with respect to the eigenfunctions $K_n(x)$, and if we also have an analytical expression for $V(x,\ \tau)$ in closed form, the problem of improving the convergence of the series (1) will be solved. The point is that the series in (2) satisfies homogeneous boundary conditions and converges uniformly over the whole range of variation of x.

The actual determination of the function $V(x,\ \tau)$ is based on a consideration of the Sturm–Liouville problem for the corresponding ordinary differential equation (which arises on constructing the kernel of

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the transformation with respect to x), and on the setting up of the corresponding Green's function (ordinary or generalized).

The form of $V(x, \tau)$ also depends on the boundary conditions of the heat-conduction problem under consideration.

For example, let the solution (1) be obtained for boundary conditions of the first kind, i.e.,

$$t(0, \tau) = \omega_1(\tau), \quad t(l, \tau) = \omega_2(\tau). \tag{3}$$

Then according to [4] the function $V(x, \tau)$ takes the form

$$V(x, \tau) = \omega_1(\tau) \frac{G'_\xi(x, 0)}{G'_\xi(0+0, 0)} + \omega_2(\tau) \frac{G'_\xi(x, l)}{G'_\xi(l-0, l)}, \tag{4}$$

where $G(x, \xi)$ is the Green's function of the boundary problem:

$$\frac{d^2K}{dx^2} + \mu^2 K = 0, \tag{5}$$

$$K(0) = K(l) = 0. \tag{6}$$

In this case $G(x, \xi)$ is given by the equation

$$G(x, \xi) = \begin{cases} \dfrac{x(l-\xi)}{l} & \text{for} \quad x < \xi, \\ \dfrac{\xi(l-x)}{l} & \text{for} \quad x > \xi. \end{cases} \tag{7}$$

Substituting (7) into (4) and making some transformations, we obtain

$$V(x, \tau) = \omega_1(\tau)\left(1 - \frac{x}{l}\right) + \omega_2(\tau)\frac{x}{l}. \tag{8}$$

On the other hand, by using the bilinear formula for the Green's function we have:

$$V(x, \tau) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [\omega_1(\tau) + (-1)^{n-1}\omega_2(\tau)] \sin \mu_n x, \tag{9}$$

where $\mu_n = \pi n/l$ is the system of eigenvalues of the problem (5), (6).

Returning to Eq. (2), we note that we may subtract the function V in the form of Eq. (9) and then add V in the form of Eq. (8). The series so obtained has a better convergence. The construction of the function V in the forms (8) and (9) is valid, since zero is not an eigenvalue of the corresponding Sturm–Liouville problem.

When one of the eigenvalues of the Sturm–Liouville problem equals zero, the construction of $V(x, \tau)$ has to be based on the generalized Green's function $G^*(x, \xi)$.

For example, let us suppose that boundary conditions of the second kind are specified in terms of the variable x:

$$\frac{\partial t(0, \tau)}{\partial x} = \omega_1(\tau), \quad \frac{\partial t(l, \tau)}{\partial x} = \omega_2(\tau). \tag{10}$$

The generalized Green's function $G^*(x, \xi)$ is in this case defined as the continuous solution of the equation

$$\frac{d^2G^*(x, \xi)}{dx^2} = K_0(x) K_0(\xi) - \delta(x - \xi) \tag{11}$$

with boundary conditions

$$G^{*\prime}(0, \xi) = G^{*\prime}(l, \xi) = 0, \tag{12}$$

where $K_0$ is the nontrivial normalized solution of the equation

**TABLE 1. Values of the Green's Function and the Function $V(x, \tau)$ for Various Combinations of Boundary Conditions**

| Type of boundary conditions | Green's function $G(x, \xi)$: bilinear formula | Green's function $G(x, \xi)$: closed form | Literature |
|---|---|---|---|
| I $t(0, \tau) = \omega_1(\tau)$ | $\frac{2l}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{l} \sin \frac{n\pi\xi}{l}}{n^2}$ | $\frac{x(l-\xi)}{l}$ | for $x<\xi$ |
| I $t(l, \tau) = \omega_2(\tau)$ | | $\frac{\xi(l-x)}{l}$ | for $x>\xi$ |
| II $\frac{\partial t(0, \tau)}{\partial x} = \omega_1(\tau)$ | $\frac{2l}{\pi_2} \sum_{n=1}^{\infty} \frac{\cos \frac{n\pi x}{l} \cos \frac{n\pi\xi}{l}}{n^2}$ | $\frac{x^2+\xi^2}{2l} + \frac{l}{3} - \xi$ | for $x<\xi$ |
| II $\frac{\partial t(l, \tau)}{\partial x} = \omega_2(\tau)$ | | $\frac{x^2+\xi^2}{2l} + \frac{l}{3} - x$ | for $x>\xi$ |
| III $\frac{\partial t(0, \tau)}{\partial x} - ht(0, \tau) = -ht_c$ | $2 \sum_{n=1}^{\infty} \frac{(\mu_n \cos \mu_n x + h \sin \mu_n x)(\mu_n \cos \mu_n \xi + h \sin \mu_n \xi)}{\mu_n^2 [l(\mu_n^2 + h^2) + 2h]}$ | $\frac{(1+hx)(1/h + l - \xi)}{2 + hl}$ | for $x<\xi$ |
| III $\frac{\partial t(l, \tau)}{\partial x} + ht(l, \tau) = ht_c$ | | $\frac{(1+h\xi)(1/h + l - x)}{2 + hl}$ | for $x>\xi$ |
| I $t(0, \tau) = \omega_1(\tau)$ | $\frac{8l}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{(2n-1)\pi x}{2l} \sin \frac{(2n-1)\pi\xi}{2l}}{(2n-1)^2}$ | $x$ | for $x<\xi$ |
| II $\frac{\partial t(l, \tau)}{\partial x} = \omega_2(\tau)$ | | $\xi$ | for $x>\xi$ |
| II $\frac{\partial t(0, \tau)}{\partial x} = \omega_1(\tau)$ | $\frac{8l}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos \frac{(2n-1)\pi x}{2l} \cos \frac{(2n-1)\pi\xi}{2l}}{(2n-1)^2}$ | $l - \xi$ | for $x<\xi$ |
| I $t(l, \tau) = \omega_2(\tau)$ | | $l - x$ | for $x>\xi$ |
| I $t(0, \tau) = \omega_1(\tau)$ | $2 \sum_{n=1}^{\infty} \frac{(\mu_n^2 + h^2) \sin \mu_n x \sin \mu_n \xi}{\mu_n^2 [l(\mu_n^2 + h^2) + h]}$ | $\frac{x[1 + h(l - \xi)]}{1 + hl}$ | for $x<\xi$ |
| III $\frac{\partial t(l, \tau)}{\partial x} + ht(l, \tau) = th_c$ | | $\frac{\xi[1 + h(l - x)]}{1 + hl}$ | for $x>\xi$ |

| Type of boundary condtions | Function $V(x, \tau)$, which is added to the solution | Function $V(x, \tau)$, which is subtracted from the solution | |
|---|---|---|---|
| II $\frac{\partial t(0, \tau)}{\partial x} = \omega_1(\tau)$ | $2 \sum_{n=1}^{\infty} \frac{(\mu_n^2 + h^2) \cos \mu_n x \cos \mu_n \xi}{\mu_n^2 [l(\mu_n^2 + h^2) + h]}$ | $\frac{1}{h} + l - \xi$ | for $x<\xi$ |
| III $\frac{\partial t(l, \tau)}{\partial x} + ht(l, \tau) = ht_c$ | | $\frac{1}{h} + l - x$ | for $x>\xi$ |
| III $\frac{\partial t(0, \tau)}{\partial x} - ht(0, \tau) = -ht_c$ | $2 \sum_{n=1}^{\infty} \frac{(\mu_n^2 + h^2) \sin \mu_n (l - x) \sin \mu_n (l - \xi)}{\mu_n^2 [l(\mu_n^2 + h^2) + h]}$ | $\frac{(1+hx)(l-\xi)}{1+hl}$ | for $x<\xi$ |
| I $t(l, \tau) = \omega_2(\tau)$ | | $\frac{(1+h\xi)(l-x)}{1+hl}$ | for $x>\xi$ |
| III $\frac{\partial t(0, \tau)}{\partial x} - ht(0, \tau) = -ht_c$ | $2 \sum_{n=1}^{\infty} \frac{(\mu_n^2 + h^2) \cos \mu_n (l - x) \cos \mu_n (l - \xi)}{\mu_n^2 [l(\mu_n^2 + h^2) + h]}$ | $\frac{1+hx}{h}$ | for $x<\xi$ |
| II $\frac{\partial t(l, \tau)}{\partial x} = \omega_2(\tau)$ | | $\frac{1+h\xi}{h}$ | for $x>\xi$ |

TABLE 1 (continued)

| Type of boundary conditions | Fuction V (x, $\tau$) which is added to the solution | Fuction V (x, $\tau$) which is subtracted from the solution | Literature |
|---|---|---|---|
| I $t(0,\tau)=\omega_1(\tau)$<br>I $t(l,\tau)=\omega_2(\tau)$ | $\omega_1(\tau)\left(1-\frac{x}{l}\right)+\omega_2(\tau)\frac{x}{l}$ | $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[\omega_1(\tau)+(-1)^{n-1}\omega_2(\tau)]\sin\frac{n\pi x}{l}$ | [4] |
| II $\frac{\partial t(0,\tau)}{\partial x}=\omega_1(\tau)$<br>II $\frac{\partial t(l,\tau)}{\partial x}=\omega_2(\tau)$ | $\frac{l}{2}\omega_2(\tau)\left[\left(\frac{x}{l}\right)^2-\frac{1}{3}\right]-\frac{l}{2}\omega_1(\tau)\times\left[\left(1-\frac{x}{l}\right)^2-\frac{1}{3}\right]$ | $\frac{2l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n\omega_2(\tau)-\omega_1(\tau)}{n^2}\right]\cos\frac{n\pi x}{l}$ | [4] |
| III $\frac{\partial t(0,\tau)}{\partial x}-ht(0,\tau)=-ht_c$<br>III $\frac{\partial t(l,\tau)}{\partial x}+ht(l,\tau)=ht_c$ | $t_c$ | $4ht_c\sum_{n=1}^{\infty}\frac{\mu_n\cos\mu_n x+h\sin\mu_n x}{\mu_n[(\mu_n^2+h^2)l+2h]}$ | This paper |
| I $t(0,\tau)=\omega_1(\tau)$<br>II $\frac{\partial t(l,\tau)}{\partial x}=\omega_2(\tau)$ | $\omega_1(\tau)+\omega_2(\tau)x$ | $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left[\omega_1(\tau)+\frac{(-1)^{n-1}2l}{(2n-1)\pi}\omega_2(\tau)\right]\sin\frac{(2n-1)\pi x}{2l}$ | " |
| II $\frac{\partial t(0,\tau)}{\partial x}=\omega_1(\tau)$<br>I $t(l,\tau)=\omega_2(\tau)$ | $\omega_2(\tau)-\omega_1(\tau)(l-x)$ | $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left[(-1)^{n-1}\omega_2(\tau)-\frac{2l}{(2n-1)\pi}\omega_1(\tau)\right]\cos\frac{(2n-1)\pi x}{2l}$ | " |
| I $t(0,\tau)=\omega_1(\tau)$<br>III $\frac{\partial t(l,\tau)}{\partial x}+ht(l,\tau)=ht_c$ | $\omega_1(\tau)\frac{1+h(l-x)}{1+hl}+\frac{ht_c x}{1+hl}$ | $2\sum_{n=1}^{\infty}\frac{\mu_n^2+h^2}{\mu_n[l(\mu_n^2+h^2)+h]}\left[\omega_1(\tau)-\frac{ht_c}{\sqrt{\mu_n^2+h^2}}\right]\sin\mu_n x$ | " |
| II $\frac{\partial t(0,\tau)}{\partial x}=\omega_1(\tau)$<br>III $\frac{\partial t(l,\tau)}{\partial x}+ht(l,\tau)=ht_c$ | $t_c-\omega_1(\tau)\left(\frac{1}{h}+l-x\right)$ | $2\sum_{n=1}^{\infty}\frac{\mu_n^2+h^2}{\mu_n[l(\mu_n^2+h^2)+h]}\left[\frac{ht_c}{\sqrt{\mu_n^2+h^2}}-\frac{\omega_1(\tau)}{\mu_n}\right]\cos\mu_n x$ | " |
| III $\frac{\partial t(0,\tau)}{\partial x}-ht(0,\tau)=-ht_c$<br>I $t(l,\tau)=\omega_2(\tau)$ | $\frac{ht_c(l-x)}{1+hl}+\omega_2(\tau)\frac{1+hx}{1+hl}$ | $2\sum_{n=1}^{\infty}\frac{\mu_n^2+h^2}{\mu_n[l(\mu_n^2+h^2)+h]}\left[\omega_2(\tau)-\frac{ht_c}{\sqrt{\mu_n^2+h^2}}\right]\sin\mu_n(l-x)$ | " |
| III $\frac{\partial t(0,\tau)}{\partial x}-ht(0,\tau)=-ht_c$<br>II $\frac{\partial t(l,\tau)}{\partial x}=\omega_2(\tau)$ | $t_c+\omega_2(\tau)\frac{1+hx}{h}$ | $2\sum_{n=1}^{\infty}\frac{\mu_n^2+h^2}{\mu_n[l(\mu_n^2+h^2)+h]}\left[\frac{ht_c}{\sqrt{\mu_n^2+h^2}}+\frac{\omega_2(\tau)}{\mu_n}\right]\cos\mu_n(l-x)$ | " |

$$\frac{d^2K}{dx^2} = 0, \tag{13}$$

satisfying the boundary conditions

$$K'(0) = K'(l) = 0. \tag{14}$$

Solving the boundary problems (13), (14) and (11), (12), and allowing for the condition of orthogonality between K and $G^*(x, \xi)$, after a few transformations we obtain

$$G^*(x, \xi) = \begin{cases} \dfrac{x^2 + \xi^2}{2l} + \dfrac{l}{3} - \xi & \text{for} \quad x < \xi, \\ \dfrac{x^2 + \xi^2}{2l} + \dfrac{l}{3} - x & \text{for} \quad x > \xi. \end{cases} \tag{15}$$

Substituting (15) into G. A. Grinberg's equation for $V(x, \tau)$ in the case of a zero eigenvalue

$$V(x, \tau) = \omega_1(\tau) \frac{G^*(x, 0)}{G_x^{*'}(0 + 0, 0)} + \omega_2(\tau) \frac{G^*(x, l)}{G_x^{*'}(l - 0, l)}, \tag{16}$$

after some calculations we obtain

$$V(x, \tau) = \frac{1}{2} \omega_2(\tau) l \left[\left(\frac{x}{l}\right)^2 - \frac{1}{3}\right] - \frac{1}{2} \omega_1(\tau) l \left[\left(1 - \frac{x}{l}\right)^2 - \frac{1}{3}\right]. \tag{17}$$

Using the bilinear equation for the generalized Green's function, by analogy with the earlier problem, we obtain an expression for V in the form of the following series:

$$V(x, \tau) = \frac{2l}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n \omega_2(\tau) - \omega_1(\tau)}{n^2} \cos \frac{n\pi x}{l}. \tag{18}$$

Here we must remember that the series forming the solution is

$$t = \bar{t}_0 K_0 + \sum_{n=1}^{\infty} \bar{t}_n(\tau) K_n(x), \tag{19}$$

where $t_0$ and $K_0$ are expressions corresponding to the zero eigenvalue (for example, if $K_n = \sqrt{(2/l)}\cos(n\pi x/l)$, then $K_0 = 1/\sqrt{l}$).

If on the right-hand side of Eq. (19) we add V in the form (17) and subtract V in the form (18), we obtain a solution in which the series has a considerably better convergence.

The foregoing method of constructing the auxiliary function in two forms may be extended to cases of various other boundary conditions. The function $V(x, \tau)$ was plotted for a few such cases in [4].

In this paper we present the function $V(x, \tau)$ for a number of combinations of boundary conditions in the Cartesian coordinate system (Table 1).

By way of example, let us consider improving the convergence of the series for the problem:

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2} + w_0(x, \tau), \quad t(x, 0) = f(x),$$
$$\frac{\partial t(0, \tau)}{\partial x} - ht(0, \tau) = -ht_c, \quad \frac{\partial t(l, \tau)}{\partial x} + ht(l, \tau) = ht_c. \tag{20}$$

The solution of the problem (20) obtained by the method of finite integral transformations takes the form

$$t = 2 \sum_{n=1}^{\infty} \frac{\exp(-a\mu_n^2 \tau)}{(\mu_n^2 + h^2) l + 2h} (\mu_n \cos \mu_n x + h \sin \mu_n x)$$
$$\times \left\{ \int_0^{\tau} \left[ \int_0^l w_0(x, \tau)(\mu_n \cos \mu_n x + h \sin \mu_n x)\, dx \right] \exp(a\mu_n^2 \tau)\, d\tau \right. \tag{21}$$
$$\left. + \int_0^l f(x)(\mu_n \cos \mu_n x + h \sin \mu_n x)\, dx + \frac{2ht_c}{\mu_n} [\exp(a\mu_n^2 \tau) - 1] \right\}.$$

Using the expression for V given in Table 1 (case III) we may improve the convergence of the series (21). After some simple transformations we obtain

$$t = t_c + 2\sum_{n=1}^{\infty} \frac{\exp(-a\mu_n^2\tau)}{(\mu_n^2 + h^2)\, l + 2h} (\mu_n \cos\mu_n x + h \sin\mu_n x)$$
$$\times \left\{ \int_0^{\tau} \left[ \int_0^{l} w_0(x, \tau)(\mu_n \cos\mu_n x + h \sin\mu_n x)\, dx \right] \exp(a\mu_n^2\tau)\, d\tau + \int_0^{l} f(x)(\mu_n \cos\mu_n x + h \sin\mu_n x)\, dx - \frac{2ht_c}{\mu_n} \right\}. \tag{22}$$

It is not hard to see that Eq. (22) coincides with the solution of the analogous problem of [5], in which the inhomogeneous equations are allowed for by using the solution of an auxiliary steady-state heat-conduction equation.

## NOTATION

| | |
|---|---|
| $t(x, \tau)$ | is the temperature; |
| $\bar{t}_n(\tau)$ | is the image function of temperature in the method of finite integral transformations; |
| $K_n(x)$ | is the system of eigenfunctions; |
| $l$ | is the thickness of plate; |
| $(0, l)$ | is the range of the variable x; |
| $\xi$ | is the point inside the range $(0, l)$; |
| $\delta(z)$ | is the Dirac delta function; |
| $a$ | is the thermal diffusivity; |
| $w_0(x, \tau) = w(x, \tau)/C$; | |
| $w(x, \tau)$ | is the density of heat sources; |
| $C$ | is the specific heat; |
| $h = \alpha/\lambda$ | is the relative heat-transfer coefficient; |
| $\alpha$ | is the heat-transfer coefficient; |
| $\lambda$ | is the thermal conductivity; |
| $t_c$ | is the temperature of the medium. |

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